Lattice Green functions and diffusion for modelling traffic routing in *ad hoc* networks

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***Abstract*—We describe basic properties of Markov chains on finite state spaces and their application to Green functions, partial differential equations, and their (approximate) solution using random walks on a graph. Attention is paid to the influence of boundary conditions (Dirichlet/von Neumann). We apply these ideas to the study of traffic propagation and distribution in *ad hoc* networks.**

I. Introduction

In *ad hoc* and sensor networks, determining traffic intensity patterns is one of the most important issues. Indeed, one of the performance measures of routing protocols is data transmission delay and throughput, which respectively depend on levels of congestion in the network and collisions occurring when too many nodes try to access the transmission medium. In fact, both congestion and collisions directly depend on the intensity of the traffic that each node is expected to relay for a given routing protocol. In general, when no congestion is present, the shortest path routing protocol provides the best performance in terms of delay, since the data travel from source to destination using the shortest path through the network. However, this protocol has been shown to lead to severe congestion and collisions problems when the traffic generated by each node is high [1], [2]. In this case, it is preferable to balance the load of traffic between source and destination by splitting the initial flow of data onto parallel paths. The design of such load balance routing has to take into account the distribution of the traffic as well as the performance in terms of delay. Moreover, whether such load balancing is actually needed depends on the level of congestion when shortest path routing is used. Hence, a method for systematically analysing the traffic patterns in the network is needed.

In their study of traffic distribution and routing in *ad hoc* networks [1], [2], Pham and Perreau proved that the traffic relayed by each node is proportional to the number of shortest paths going through this node. Then they showed that for a circular network with radius R, the number of such paths for a node $\mathbf{y}(r)$ located at a distance r from the centre of the network is

$$\frac{\pi}{2}\ \rho^2\ \beta\ (R^2 - r^2)^2\ , \tag{1}$$

where ρ denotes the (uniform) spatial density of nodes and β is a (small) allowed deflection angle wrt. the shortest path (see Fig. 1).

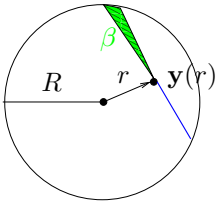

Fig. 1. A circular lattice zone

While this expression is a reasonable approximation of the traffic pattern, the deflection angle can only be determined heuristically. Moreover, the methodology does not extend to load balancing routing protocols, so that their performance is difficult to predict with accuracy.

In this paper, we generalize the results of [1], [2] using the formalism of Green functions. (Further details can be found in [3].) In sections II and III, we recall how to find approximate solutions to PDEs using diffusion walks on discrete graphs (lattices) for various boundary and source conditions, and note that the solutions known as discrete Green functions count the number of paths between two nodes under these diffusion processes. In section IV, we follow a parallel continuous analysis, and thereby reproduce the result of [1], [2] in this more general setting.

II. Discretization of a class of linear (elliptic) PDEs

We consider a **finite** state space $\mathcal{E}$ with cardinality N, typically a (regular) subset of $\mathbb{Z}^d$, and the vector space $E = \mathbb{R}^{|\mathcal{E}|}$ of real **bounded functions** mapping $\mathcal{E}$ to $\mathbb{R}$:

$$x \mapsto f(x)$$

whose canonical basis vectors are functions $e_y(x) = \mathbb{1}_{x=y}$.

Following [4], [5], we search for $f \in E$ such that:

$$\left\{ \begin{array}{ll} \gamma\ (A - \mathrm{I})\ f = \phi\ , & \phi \in E\ ,\ A \in \mathcal{L}(E) \\ (eventually)\ subject\ to & \\ f(x_s) = b(x_s)\ \text{known} & \forall x_s \in \mathcal{B} \subset \mathcal{E} \end{array} \right. \tag{2}$$

Here γ is a constant, and $\mathcal{B}$ represents the boundary of domain $\mathcal{E}$ viewed as a d-dimensional discrete lattice.

A. Basic example: 2D Laplace and Poisson equations

$\mathcal{E}$ is now a discrete regular sublattice of $\mathbb{Z}^2$ with lattice step h. It is endowed with an undirected, connected graph structure, where the neighbourhood relationship is denoted $\mathbf{x_t} \sim \mathbf{x_s}$, *e.g.* 4-connectivity. The Laplace-Beltrami operator, defined as

$$\Delta f(x_s) = (\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})(x_s)$$

has the following discrete approximation:

$$\Delta f(x_s) \approx \frac{[\sum_{x_t \sim x_s} f(x_t)] - 4 f(x_s)}{h^2} .$$

Thus $\Delta f \approx \gamma(A - I)f$ with $\gamma = \frac{4}{h^2}$ and $(Af)(x_s) = \frac{1}{4}[\sum_{x_t \sim x_s} f(x_t)]$, $\forall x_s \in \mathcal{E}$. In other words, A is an **averaging operator**. It is important to note that the associated matrix, whose elements are $A_{x_s\, x_t} = \frac{1}{4}\mathbb{1}_{x_s \sim x_t}$ is a **stochastic** matrix (on $\mathcal{E}$).

B. Boundary conditions: a linear algebra point of view

Splitting $\mathcal{E}$ into boundaries and non-boundaries, the following decomposition holds:

$$\begin{array}{lll} \mathcal{E} & = \tilde{\mathcal{E}} \cup \mathcal{B} & \text{(state space)} \\ E & = \tilde{E} \oplus B & \text{(functional vector space)} \\ f & = \tilde{f} + b & \text{(functions)} . \end{array}$$

The problem (2) can thus be written:

$$\gamma\ (A - \mathrm{I})\ \tilde{f} = \phi + \gamma\ (I - A)\ b \qquad with \qquad A \in \mathcal{L}(E) .$$

Let us denote by $\tilde{P}$ the linear projector on $\tilde{E}$ with kernel $\mathcal{B}$. One has then: $\tilde{f} = \tilde{P}f = \tilde{P}\tilde{f}$ and $\tilde{P}\ b = 0$. Left-application of $\tilde{P}$ to previous equation yields:

$$\boxed{\gamma\ (\tilde{A} - \mathrm{I})\ \tilde{f} = \psi = \tilde{P}\phi - \gamma\ \tilde{P}A\ b} \qquad (3)$$

with $\tilde{A} = \tilde{P}A \in \mathcal{L}(\tilde{E})$. Also note that $\tilde{P}A\tilde{P} = \tilde{A}\tilde{P}$ and $\tilde{P}\phi$ are the restrictions of A (resp. ϕ) to $\tilde{E}$. For the Laplace-Beltrami operator, one obtains:

$$\gamma\ \tilde{P}A\ b(x_s) = \sum_{\substack{x_t \sim x_s \\ x_t \in \mathcal{B}}} b(x_t) \qquad \forall x_s \in \tilde{\mathcal{E}}.$$

In the sequel, A will often be a **stochastic** matrix (*cf.* the Poisson-Laplace case) so that $\mathbf{1}$ is an eigenfunction of A with eigenvalue $\lambda_1 = 1$. For a connected lattice [6] the multiplicity of λ_1 is 1 and all other eigenvalues verify $|\lambda_i| < 1$.

If the operator $\mathrm{I} - A$ (resp. $\mathrm{I} - \tilde{A}$) were **invertible** (assumption (a)), then

$$\tilde{f} = -\frac{1}{\gamma}\ (\mathrm{I} - \tilde{A})^{-1}\ \psi\ , \qquad (4)$$

meaning that the the 'discrete' **Green function** $G_{x\,y}$ *i.e.* the solution of (2) for $\phi(x) = \frac{\mathbb{1}_{x=y}}{h^2} = \frac{e_y}{h^2}$, would have the following closed form for null boundary conditions $b = 0$:[1]

$$G_{x\,y} = -\ \frac{1}{4}\ [(\mathrm{I} - \tilde{A})^{-1} e_y\ ](x) = -\ \frac{1}{4}\ [(\mathrm{I} - \tilde{A})^{-1}]_{x\,y} \qquad (5)$$

By the previous argument ($|\lambda_i| < 1$), this expression could then be expanded:

$$G_{x\,y} = -\frac{1}{4}\Big(\sum_{M \geq 0} (\tilde{A}^{\ M})_{x\,y}\Big) x,\ y \in \tilde{\mathcal{E}}\ . \qquad (6)$$

If, moreover, A (resp. $\tilde{A}$) were **stochastic** (assumption (b)), then one could write:

$$\gamma\ \tilde{f} = -\sum_{M=0}^{+\infty} (\tilde{A})^M\ \psi = -\sum_{M=0}^{+\infty} \mathbf{E}\,[\,\psi(X_M)\,]\ , \qquad (7)$$

where the expectation should be taken wrt. the Markov chain with transition matrix A (or $\tilde{A}$), *i.e.* the 'isotropic' 2D discrete random walk in the plane (Fig. 2)[2] This was known as early as [7], [8]. However, assumptions (a) and (b) do not always hold. Conditions for the convergence of the series (6) will be given in the sequel, based on a Markov Random Field (MRF) analysis.

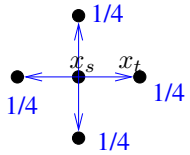

Fig. 2. Isotropic discrete random walk in the plane.

C. Laplacian on graphs and Gaussian Markov Random Fields

The (quadratic) energy functional associated to the Laplace-Beltrami operator is [6]:

$$U(f) = \sum_{x_s \sim x_t} a_{s,t}(f(x_s) - f(x_t))^2\ ,$$

where the **positive** weights $a_{s,t}$ are denoted by a comma to emphasize that they are **symmetric** (most often they are equal to 1 in our case of interest, *e.g.* Laplace-Poisson). This corresponds of course to a **Gaussian** Markov Random Field.

1) Dirichlet boundary conditions: Let us minimize $U(f)$ subject to $f(x_s) = b(x_s) \quad \forall x_s \in \mathcal{B}$. This corresponds to solving problem (3) with $\phi = 0$, and one thus finds:

$$\begin{cases} (1 - \tilde{A})\tilde{f} & = \tilde{P}\ A\ b \qquad and \\ \tilde{A}_{st} & = \dfrac{a_{s,t}}{\sum_{x_t \sim x_s} a_{s,t}} \quad \Rightarrow A \text{ is a Laplace-like operator.} \end{cases}$$

[1] We divide unit function e_y by unit cell size h^2 since it yields a 'discrete approximation' of the δ distribution. This can be seen for instance by the fact that: $\sum_{y \in \mathcal{E}} \frac{e_y}{h^2} h^2 = 1$. Notice also that h disappears in following discrete linear equations due to the particular case of a 2nd degree PDE in 2D.

[2] Recall that $\psi = \tilde{P}\phi - \gamma\ \tilde{P}A\ b$ (see (3)).

The boundary conditions of this Laplace-like problem correspond to **absorption** (see subsection III). In this case the matrix $\tilde{A}$ is both **symmetric** and **semi-stochastic**:

$$\sum_t |\tilde{A}_{st}| < 1 \text{ when } \exists\, x_t \sim x_s \; s.t. \; x_t \in \mathcal{B} i.e. \text{ near boundaries},$$

and has also no stable subspace. Under these conditions, a theorem by [9] shows that $\mathrm{I} - \tilde{A}$ is invertible and thus that the series expansions (6) and (7) do converge.

2) Von Neumann boundary conditions: To each node $x_s \in \mathcal{B}$, we associate a new node x_s^0 connected to s only (Fig. (3)). We denote the related set of sites (states) by: $\tilde{\mathcal{B}} = \{x_s^0\}_{x_s \in \mathcal{B}}$.

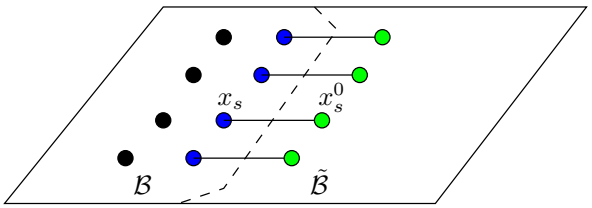

Fig. 3. Von Neumann boundary conditions.

The energy function on the extended lattice is now defined as:

$$U(f) = \sum_{x_s \sim x_t} a_{s,t}(f(x_s) - f(x_t))^2 + \epsilon \sum_{s \in B} (f(x_s) - f(x_s^0))^2 \; ,$$

with $\epsilon > 0$. Minimizing wrt. x_s^0 yields:

$$f(x_s^0) = f(x_s) \; , \tag{8}$$

which is the discrete analogue of the Von Neumann condition: $(\frac{\partial f}{\partial \vec{n}})_{x_s} = 0$. The positive coefficient ϵ is chosen such that $\epsilon \ll \min_{x_s \sim x_t} a_{s,t}$, in order not to perturb the optimal value at sites $x_s \in \mathcal{B}$, and thus:

$$f(x_s) = \frac{\sum_{x_t \sim x_s} a_{s,t} f(x_t) + \epsilon f(x_s^0)}{\sum_{x_t \sim x_s} a_{s,t} + \epsilon} \; . \tag{9}$$

In this case the matrix A is **positive stochastic** and **non-symmetric**. Indeed from (8) one has:

$$A_{x_s^0 x_s} = 1 \quad \text{and} \quad A_{x_s x_s^0} = \frac{\epsilon}{\sum_{x_t \sim x_s} a_{s,t} + \epsilon} \ll 1 \quad \forall x_s \in \mathcal{B} \; .$$

Under these conditions, a theorem by [10] based on Markov chain convergence proves that if problem (3) has a solution, then the series expansion: $\sum_{M \geq 0} (A^M)\ \psi$ converges to it (although the matrix series itself may not converge).

III. Interpretation in terms of Markov chains

We saw in previous sections that in many cases the matrix A is **stochastic** and hence the generic term in the series $(A^M)_{x\,y}$ can be interpreted as a transition probability for the Markov chain with transition matrix A. Since we need to consider $\tilde{A}^M$, as well as introduce absorbing states (see afterwards: Dirichlet conditions), it might be dangerous to modify $\tilde{A}$ in order to tailor it to a given application. The best procedure is described by [5] and [4]: design a specific Markov chain[3] with initial probability P_0 adapted to the solution, and whose **stochastic** transition matrix Q is as close as possible to $\tilde{A}$.

For instance consider the problem (3). The solution being f, and given a measure μ on $\mathcal{E}$, we want to evaluate:

$$\mu(f) = \sum_{x_0 \in \mathcal{E}} \mu(x_0)\ f(x_0) \; .$$

We consider for this purpose a Markov chain on $\mathcal{E} \cup \{\mathrm{a}\}$, where a is a new, **absorbing** state. This chain being specified by (P_0, Q), we consider the random variable

$$Z' = \frac{\mu(x_0)}{P_0(x_0)} \left(\prod_{X_{M+1} \neq a} \frac{\tilde{A}_{X_M\, X_{M+1}}}{Q_{X_M\, X_{M+1}}} \right) \frac{\psi(X_\tau)}{Q_{X_\tau\, a}} \; . \tag{10}$$

It must be understood here that $\tau + 1$ is the **hitting time** to absorbing state a starting from x_0 *i.e.*

$$X_{M+1} \neq X_{\tau+1} = a \quad \forall M < \tau \; .$$

The theory states that whatever the Markov chain considered,

$$-\mu(f) = \mathbf{E}\,[\,Z'\,] \quad w.r.t. \text{ the chosen Markov chain},$$

where the sign arises from (6). This expression can be approximated by averaging the empirical values of Z' over N^s simulations of the Markov chain prescribed by (P_0, Q):

$$-\mu(f) = \left(\sum_{i=1}^{N^s} Z'^{(i)} \right) / \; N^s \; .$$

In practice one chooses

$$\begin{cases} P_0(x_0) & = \mu_{x_0} \\ Q_{x_s\, x_t} & = \tilde{A}_{x_s\, x_t} \quad x_s, x_t \in \tilde{\mathcal{E}} \\ Q_{y\, a} & = 1 \end{cases} \; .$$

The only difference is that a is an absorbing state for this Markov chain, *e.g.* linked to a Dirichlet boundary value or to a data node, so that finally (recall that ψ is defined in (3)):

$$Z' = \psi(X_\tau) \quad s.t. \quad X_{\tau+1} = a \; .$$

The solution is thus estimated by the empirical average of values at nodes (states) connected to the absorbing state a. [4]

[3] This is one of the first instances of sequential importance sampling [11].

[4] It is important to note the **backward** aspect of these equations: for instance, y is treated as a sink here whereas it is obviously a 'source'. This relates to backward Kolmogorov-Chapman *vs.* Fokker-Planck forward equations.

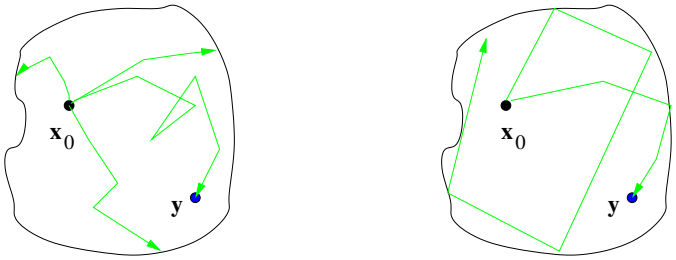

Fig. 4. Stochastic paths for various boundary conditions. Left: Dirichlet; right: von Neumann.

IV. Application to path routing in *ad hoc* networks

We show in this section that the previous results are related to modelling the number of paths arriving at a given point in a regular *ad hoc* network. During the course of this work we were not aware of the excellent work of [12]–[14]. Interesting developments on *ad hoc* networks are also found in [15], [16]. An application to Internet page finding is found in [17].
In [18]–[20] the authors define a load density vector field and minimize its total quadratic (L^2) dispersion subject to a divergence conservation law by analogy with electrostatics. A recent survey of related works is detailed in [21].

A. Two main applications of the discrete Poisson problem

We consider two applications of preceding results.

1) Markov Chain simulation of $G_{x_0\,y}$: This corresponds to choosing $\mu(x) = \mathbb{1}_{x=x_0}$, and thus to the following:

- $P_0(x) = \mu(x) = \mathbb{1}_{x=x_0}$
- The 'sink' node y is assigned to the absorbing state a
- The boundary conditions:
  - von Neumann: since the optimal solution should satisfy $f(x_s^0) = f(x_s) \ \ \forall x_s \in \mathcal{B}$, we assign
$$Q_{x_s^0, x_s} = 1 \ \ \forall x_s \in \mathcal{B} \ \ (\textbf{reflection})$$
  - Dirichlet (null / non null): we assign the absorbing state a to each $x_s \in \mathcal{B}$ (**absorption**).

From (10), one finds that $Z' = \mathbb{1}_{X_M=y \ | \ X_0=x_0}$, so that the discrete Green function (4) for null Dirichlet boundary conditions satisfies:

$$\begin{aligned} -4\ G_{x_0\,y} &= \sum_{M\geq 0} \mathbf{E}\left[\ \mathbb{1}_{X_M=y \ | \ X_0=x_0}\ \right] \\ &= \mathbf{E}\left[\ \mathbb{1}_{exists\ M\geq 0\ s.t.\ X_M=y \ | \ X_0=x_0}\ \right] \\ &= Pr\ (\exists \text{ path: } x_0 \ \rightsquigarrow \ y) \quad !! \end{aligned}$$

The last expression can be approximated by its empirical value:

$$Pr\ (\exists \text{ path: } x_0 \ \rightsquigarrow \ y) \approx \frac{N_{x_0,y}}{N_{x_0}}\ ,$$

where $N_{x_0,y}$ counts, among the N_{x_0} simulated random paths with origin x_0, the number of paths that arrive at y.

Hence the discrete approximation of the Green function for the continuous Poisson problem

$$\Delta_x G(x,y) = \delta(x,y)$$

with null Dirichlet absorbing boundary conditions counts, up to factor and sign, the number of paths from x to y.[5]

2) Markov Chain simulation of $\mu_0(G)$: Another application consists of assigning $P_0 = \mu_0$, *i.e.* the starting point of the Markov chain is chosen at random uniformly in $\mathcal{E}$. Since the solution of the Poisson problem is $f(x_0) = G_{x_0\,y} \ \ \forall x_0 \in \mathcal{E}$, this yields:

$$\mu_0(f) = -\frac{\sum_{x_0\in\mathcal{E}} G_{x_0\,y}}{(\ N = |\mathcal{E}|\ )} = \frac{1}{4}\ Pr\ (\exists \text{ path} \ \ \rightsquigarrow \ y) \approx \frac{1}{4}\ \frac{N^y}{N^s}\ ,$$

where N^y counts the number of paths arriving at the (absorbing) state y out of the N^s simulated paths starting uniformly at random in $\mathcal{E}$.

B. A continuous formalism to find the number of paths passing through some point inside a disk D

We shall compute this number using the *continuous* framework, starting from the third Green-Ostrogradsky formula in a compact domain $D \subset \mathbb{R}^2$ (with bold notation $\mathbf{x}, \mathbf{y} \in D$):

$$\begin{aligned} &\iint_D \left(F(\mathbf{x})\ \Delta G(\mathbf{x}) - G(\mathbf{x})\ \Delta F(\mathbf{x})\right)\ \mathrm{d}\mathbf{x} \\ &= \int_{\partial D} \left(F(\mathbf{x})\ \vec{\nabla} G(\mathbf{x}) - G(\mathbf{x})\ \vec{\nabla} F(\mathbf{x})\right)\ .\ \vec{n}\ \mathrm{d}s(\mathbf{x})\ . \end{aligned} \quad (11)$$

Now the Green function of the Laplace operator, defined as:

$$\Delta_{\mathbf{x}} \mathcal{G}(\mathbf{x},\ \mathbf{y}) = \delta(\mathbf{x},\ \mathbf{y}) \ \ \forall \mathbf{x},\ \mathbf{y} \in D\ , \quad (12)$$

with the 'distribution' $\delta(\ ,\ \ )$ meaning that

$$\iint_D F(\mathbf{x})\ \Delta_{\mathbf{x}} \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} = F(\mathbf{y}) \ \ \forall \mathbf{y} \in D \ \ \forall F \in \mathcal{F}$$

(where $\mathcal{F}$ is some function space) is such that any solution $F(.)$ of the Poisson problem with null boundary conditions:

$$\begin{aligned} &\Delta F = u \text{ with } F(\mathbf{x}) = 0 \ \ \forall \mathbf{x} \in \partial D \text{ (Dirichlet)} \quad (13) \\ &\textit{i.e.}, \mathcal{G}(\mathbf{x},\ \mathbf{y}) = 0 \ \ \forall \mathbf{x} \in \partial D, \forall \mathbf{y} \in D \end{aligned}$$

satisfies the following:

$$\begin{aligned} F(\mathbf{y}) \ \ &= \iint_D \Delta F(\mathbf{x})\ \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} \\ &= \iint_D u(\mathbf{x})\ \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x}\ . \end{aligned} \quad (14)$$

The 'propagator' aspect of Green functions can be seen here in the sense that they convey 'information' from point $\mathbf{x}$ to point $\mathbf{y}$ via the factor $\mathcal{G}(\mathbf{x},\ \mathbf{y})$.

Let us now use this to prove Perreau and Pham's formula above (1), by computing the number of paths passing through a node $\mathbf{y}$ in a disk D with uniform density of mobiles ρ. This number is:

$$\mathcal{N}_{\mathbf{y}} = \iint_D \mathcal{N}_{\mathbf{xy}}\ \mathrm{d}\mathbf{x} \times \iint_D \mathcal{N}_{\mathbf{yz}}\ \mathrm{d}\mathbf{z}\ ,$$

[5] Notice that the first terms $M < ||x_0 - y||_1$ (the L^1 norm) are null since no transitions occur from x_0 to y in less than $||x_0 - y||_1$ steps.

where $\mathcal{N}_{\mathbf{xy}}$ (resp. $\mathcal{N}_{\mathbf{yz}}$) are the number of paths with origin $\mathbf{x}$ and destination $\mathbf{y}$ (resp. origin $\mathbf{y}$ and destination $\mathbf{z}$.) Suppose, following the previous discrete analysis, that this number is given by

$$\mathcal{N}_{\mathbf{y}} = \int\int_D \rho\ \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \ \mathrm{d}\mathbf{x} \times \int\int_D \rho\ \mathcal{G}(\mathbf{y},\ \mathbf{z})\ \mathrm{d}\mathbf{z}\ .$$

Then

$$\mathcal{N}_{\mathbf{y}} = \rho^2\ \left(\int\int_D \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x} \right)^2\ ,$$

(the Green function is symmetric). Now consider the quantity

$$\chi(\mathbf{y}) = \int\int_D \mathcal{G}(\mathbf{x},\ \mathbf{y})\ \mathrm{d}\mathbf{x}\ .$$

From (14), $\chi(.)$ is the solution of the Poisson equation

$$\Delta\chi(\mathbf{y}) = 1 \ \ \text{with} \ \ \chi(\mathbf{y}) = 0 \ \ \forall \mathbf{y} \in \partial D\ .$$

The solution is

$$\chi(\mathbf{y}) = \frac{1}{4} \left(\ |\mathbf{y}|^2 - R^2 \right)\ ,$$

since $\Delta(x^2 + y^2) = 4$. Thus

$$\mathcal{N}_{\mathbf{y}} = (\rho\ \chi(\mathbf{y}))^2 = \frac{1}{16}\ \rho^2\ \left(\ |\mathbf{y}|^2 - R^2 \right)^2\ .$$

Up to a multiplicative constant, this is the same as (1) from [1], [2].

V. Conclusion

We have presented an application of Markov chains and diffusion processes on graphs and lattices to the calculation of traffic density in *ad hoc* networks, generalizing the expression computed in [1], [2] to Brownian paths. While it is not surprising that any first-order Markov chain would possess the same behaviour in the continuum limit, it is perhaps more surprising that the same behaviour arises in this case as in [1], [2], since the paths allowed there are a singular case of a second-order chain. The next steps are:

- to look at von Neumann boundary conditions, since it is more likely that the traffic is reflected at boundaries;
- to establish a 'flow equation' that more realistically represents the traffic routing mechanism between two points in a given network, and to derive the 'ensemble' equations associated to all pairs of such nodes;
- to study loop-erasing walks [22], since admissible traffic routing paths should contain no loops.

Acknowledgements

The authors thank Xavier Lagrange (ENST Bretagne) for providing useful information before publication; Olivier Cappé and Gersende Fort (Telecom ParisTech) for essential discussions about Markov chains; Jérôme Darbon (UCLA) for his helpful comments and suggestions; and the referees for essential, meaningful references.

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